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# Contents

Chapter One

# "Cardinality Equals Variety for Chords" in Well-Formed Scales, with a Note on the Twin Primes Conjecture

David Clampitt

## 1.1 Scope, Method, and Aim of Scale Theory

Researchers have, in the past several decades, used formal approaches to diatonic theory in an attempt to show why the features of certain pitch collections have had such appeal for composers.[1] The results relate either to what musicians have discovered they can do with a given collection—through moves, routines, or processes within the collection, or through manipulation of the collection itself—or to how a given collection functions cognitively, based upon measures of symmetry versus asymmetry, simplicity versus complexity, or information versus redundancy.[2]

Investigations of the first type use transformational theory and analysis. For example, harmonic triads, and the usual pentatonic and diatonic sets, all participate in maximally smooth cycles, the starting-point for neo-Riemannian analysis. Triads in an octatonic setting, or dominant and half-diminished seventh chords in non-maximally smooth settings, also suggest a neo-Riemannian approach; certain pitch-class sets in an atonal setting allow for analogous procedures using transformational analysis.[3]

Investigations of the second type tend to be too general to apply to analysis, but help explain the popularity of certain systems—always within particular cultural contexts, be it understood. Such explanations are by no means the exclusive province of self-described diatonicists and other scale theorists. Carl

Dahlhaus offers a diatonicist distinction between Guido's hexachord, on the one hand, and the usual pentatonic and diatonic, on the other: the latter are "systems," the former "is not, in contrast to the heptatonic and pentatonic scales, a self-significant system of tones," but is "a mere auxiliary construction."[4] What Dahlhaus means is that in the pentatonic and diatonic one always understands which intervals are steps and leaps, relative to the system: "minor thirds" in the pentatonic are always "steps," while in the diatonic they are always "leaps," as opposed to the situation in the hexachord, where "the listener would have to alternate between . . . the idea of the minor third as a 'step' and as a 'leap'."[5] He labels this an "absurd consequence." The vehemence of Dahlhaus's language here must be understood within the context of a particular historical/theoretical discussion, on the development of the tonal system. But the implicit assumption, that a simpler relationship between intervals and their description in terms of scale steps provides for smoother cognitive processing, has more general applications. Taking Dahlhaus's observation and considering it closely (nowhere does he expressly say that all three of these entities are generated by the perfect fifth, though this is the case), and formulating it in mathematical terms, one is led to the concept of a "well-formed scale," defined by Norman Carey and myself and discussed later in this article.

These two types of properties—we might call them "operational" and "systemic"—are not entirely separable: the transformational property of the diatonic that allows one to modulate in a maximally smooth way between fifth-related sets, by moving a single note by a minimal distance (a chromatic semitone), is an essential aspect of a fully developed tonal system. Both types of properties lend themselves to mathematical investigation, one of the hallmarks of scale theory. The aim of the mathematical formalization and generalization of such properties is to understand how they work and how they relate to each other: for example, what is the relationship between the fact that a given scale admits a generating interval, and the number and arrangement of the scale's various step intervals? A mathematically derived result is that in such cases step intervals come in at most three sizes (as exemplified in the three entities—pentachordal, hexachordal, and heptachordal—discussed by Dahlhaus).[6]

The subject of this paper, the property "cardinality equals variety," to be defined, is a systemic property that assumes that describing some musical objects both in terms of intervals and in terms of scale steps is useful. In particular, that the number of varieties of such objects, such as chords or pitch-class lines, matches the number of distinct constituents of these objects, is hypothesized to be advantageous. Along the axis of information and redundancy, "cardinality equals variety" entails a system that is not too exciting, not too boring, and that has a high degree of organization. "Cardinality equals variety" also forms the starting point for Johnson's textbook, which attempts to be at least a partial answer to Cohn's call for a "new pedagogibility" in music theory.[7]

## 1.2 Cardinality Equals Variety for Chords

In their 1985 *Journal of Music Theory* article "Variety and Multiplicity in Diatonic Systems," Clough and Myerson showed that MP $\Leftrightarrow$ CV for lines; that is, any pitch-class set with Myhill's Property (in chromatic universes of any cardinality) has cardinality equals variety for lines, and (trivially) vice versa.[8] The basic notion these concepts depend upon is the distinction between generic and specific interval measures (diatonic and chromatic lengths, respectively). Generic measure is simply the measure musicians commonly use when they take an assumed underlying diatonic set as a ruler, and speak of "seconds" or "thirds" or "sixths," without regard to quality. The appropriate mathematics for discussions of generic measure in the usual diatonic set is arithmetic mod 7.[9] Specific measure, on the other hand, deals with the actual quality of the interval. In Clough and Myerson's approach, the diatonic is assumed to be embedded within a mod 12 chromatic universe; thus, specific interval measure is just the usual mod 12 measure of musical set theory. More generally, they apply generic and specific interval measures in chromatic universes of arbitrary cardinality $c$, and in pitch-class subsets of cardinality $d < c$.

Clough and Myerson also categorize unordered and ordered pitch-class sets, i.e., chords and lines, according to the generic/specific distinction. Unlike the practice in traditional musical set theory, however, Clough and Myerson's equivalence classes use transposition only, not inversion. Thus, the unordered pitch-class subsets (chords) in the C-major diatonic set {C, D, F} and {D, F, G} are considered to be non-equivalent, even though they map onto each other under inversion under both mod 7 and mod 12 measures. Chord types are displayed using interval normal form.[10] The trichord {C, D, F} is a member of generic class (124) (i.e., step, third, fifth), and specific class (237), (i.e., major second, minor third, perfect fifth), while the trichord {D, F, G} is a member of generic class (214) and specific class (327). The C-major subset {F, A, B}, on the other hand, is also a member of generic class (214), but with respect to specific measure is of type (426). As a line, the ordered set ⟨F A B⟩ is described generically as ⟨2 1 4⟩, and specifically as ⟨4 2 6⟩.

A set is said to have Myhill's Property (MP) if every non-zero generic interval comes in two specific varieties (with "non-zero" understood to include "non-octave equivalent"). The usual diatonic is an example of a set with MP. A set has cardinality equals variety (CV) for lines if any line of a given generic description comes in $k$ specific varieties, where $k$ is the number of distinct pitch classes in the line. Again, the usual diatonic is an example, as is the complementary pentatonic. Diatonic seventh chords, for instance, come in four varieties: major, minor, dominant, and half diminished; arpeggiated, these form four different types of pitch-class lines. Since an interval may be construed as a two-note line, if a set has CV for lines, it automatically has MP. Clough and Myerson proved the non-trivial converse, that any set that has MP exhibits CV for lines.

As Clough and Myerson pointed out, however, the situation regarding CV for chords is more delicate. First of all, if *c* is the cardinality of the chromatic universe and *d* is that of a pitch-class subset that has MP, while CV for lines holds for lines with *d* distinct pitch classes, there is clearly only one chord containing all *d* pitch classes. In the usual diatonic, for example, the seven line species corresponding to the generic line class ⟨1 1 1 1 1 1 1⟩ are the seven diatonic modes, whereas the 7-pc chord is just the diatonic aggregate, of specific description (1 2 2 1 2 2 2). Thus, one must exclude the set as a whole from the discussion, and say that a set exhibits CV for chords if for all integers *e* such that $1 \leq e < d$, every chord of cardinality *e* of a given generic description comes in *e* specific varieties. In the usual diatonic and pentatonic scales, CV for chords holds, but Clough and Myerson gave a counterexample of an MP set where CV for chords fails. This occurred in a case where the cardinality of the chord is not coprime with *d*, the cardinality of the MP set. Clough and Myerson concluded with the following conjecture: "It seems that the best one can say is that, if a scale has MP, then CV holds except for certain chords of cardinality not coprime with *d*."[11]

The purpose of this essay is to determine precisely under what conditions CV for chords holds. I will consider a slightly more general setting: while all the MP sets in Clough and Myerson are embedded (that is, subsets of a finite chromatic universe), I will also consider non-embedded sets, wherein the sizes of the constituent intervals may be incommensurable with the size of the octave or other modular unit. For example, in the diatonic scale in Pythagorean tuning or in quarter-comma mean-tone temperament, the sizes of the non-zero intervals (those other than unison or its octave equivalents) are irrational values if the octave itself is taken as the unit of measure. Hence, it is impossible to express these scales as subsets of an equal division of the octave. Nonetheless, it is still possible to invoke the generic/specific distinction, and to say that the diatonic scale in either of these two tunings exhibits a slightly generalized Myhill's Property. To do so, we simply adopt the notion of generic measure already given, and generalize the notion of specific measure to cover the cases of irrational divisions of the octave.

As Carey and I have shown, sets with generalized MP are equivalent to nondegenerate well-formed sets.[12] A set is well-formed if it admits a generator, where that generator is spanned by a constant number of scale-step intervals. For example, the usual diatonic is generated by the perfect fifth (or perfect fourth), and in the diatonic set, the perfect fifth is always spanned by 4 step intervals (and the perfect fourth is always spanned by 3 step intervals). While it may sound tautological to say that a fifth in the diatonic set is always a fifth, that is only because we customarily use diatonic nomenclature. If we consider the hexachord generated by perfect fifths or fourths, we see that there are some "perfect fifths" that are spanned by 4 steps, and others that are spanned by 3 steps (within the hexachord).

Any equal division of the octave trivially satisfies the definition of a well-formed scale: if $U_c$ is a chromatic universe of cardinality $c$, then any interval represented by the integer $h$ where $h$ is coprime with $c$ will generate the whole set and will be spanned by a constant number of steps. Setting $h = 1$ and generating all $c$ elements will always work, for example. Carey and I call these *degenerate* well-formed sets, and they obviously do not have MP. Hereinafter, whenever I write "well-formed" I intend "*non-degenerate* well-formed," and given the equivalence mentioned above, "generalized MP" may always be substituted for "well-formed."

The results on CV for chords in well-formed sets, to be demonstrated below, are as follows: CV for chords holds in D if and only if D is a non-degenerate well-formed set of cardinality $d$ where $d$ is prime and where if D is embedded in a chromatic universe of cardinality $c$, $d \leq \lfloor c/2 \rfloor + 1$.[13] Furthermore, if $d > \lfloor c/2 \rfloor + 1$, CV fails for a chord of cardinality $e = 2d - c$, and if $d$ is composite, CV fails for chords of cardinality $e$ where the greatest common divisor of $d$ and $e$ is $n > 1$. The proofs of these statements are given later in this paper.

Before considering the mathematical treatment of CV for chords in general, it may be useful to observe the situation in a concrete case, involving a familiar musical object of small cardinality, the usual pentatonic scale.

The pentatonic case is displayed in table 1.1. For now, we are just looking at the first three columns. The third column shows the literal unordered subsets of the pentatonic scale with pitch classes C D F G A, except for the bottom box, which gives the five different scalar orderings of the pentatonic scale (the five "modes"). The first column displays the pentatonic genera, up to pitch-class equivalence; the second column partitions each of the genera into species, each of which is measured by the semitone of 12-tone equal temperament. The top two boxes give the dyads, or two-note literal subsets: they come in two genera, each of which contains two species. Observe that genera are given by partitions of 5 and species by partitions of 12—this is the Clough-Regener notation introduced above in the case of the usual diatonic set. Once we have checked CV for the dyads and find that Myhill's Property holds, we know from Clough and Myerson's result that CV for lines holds for the pentatonic. The rest of the table's first two columns verifies this for CV for chords (i.e., for unordered subsets of cardinalities 2–4), a stronger property than CV for lines (even though there are an infinite number of instantiations in pitch space of CV for lines).

Columns 4 and 5 use a tool from the foundations of well-formed scale theory: the generating interval of constant span. This defining characteristic implies that one can use the generating interval as a measure of both specific and generic intervals. Consider again the familiar diatonic case. All specific diatonic intervals modulo the octave may be represented by some number, positive or negative, of perfect fifths: thus, +2 fifths (reduced by an octave) is the ascending whole step. That is, +2 represents all directed pitch-class intervals of the type C–D, D–E, F♯–G♯, etc. The rising semitone E–F, on the other hand, is represented in perfect fifths measure by −5 (moving counterclockwise on the diatonic circle of fifths, if

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